Special values of Rankin-Selberg automorphic L-functions for $\mathrm{GL}_n \times \mathrm{GL}_m$ over totally real fields

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Giancarlo Castellano

Fordham University

gcastellano7@fordham.edu homepage.univie.ac.at/giancarlo.castellano/

Special L-values and periods

Starting point

- A Rankin–Selberg automorphic L-function $L(s, \pi \times \pi')$ for $\mathrm{GL}_n \times \mathrm{GL}_m$, where $m < n$, over a number field F.
- A *critical point* for $L(s, \pi \times \pi')$ is an $s_0 \in \frac{n-m}{2} + \mathbb{Z}$ such that the gamma factors on both sides of the functional equation are holomorphic at s_0.

Goal Understand the arithmetic of the values $L(s_0, \pi \times \pi')$ and of quotients $\frac{L(s_0, \pi \times \pi')}{L(s_0+1, \pi \times \pi')}$ when π and π' are *regular algebraic* (equivalently, *cohomological*).

Main ingredient: Automorphic periods

Periods of a regular algebraic representation π of GL_n arise from comparing independently defined rational structures on different *models* of π.

- *Whittaker–Betti periods* $p^{\varepsilon}(\pi)$: compare the Whittaker model and the bottom-degree cohomological model $H^{b_n^F}(S^{\mathrm{GL}_n}, \widetilde{\mathcal{E}}_\lambda)(\pi_{\mathrm{f}} \times \varepsilon)$.
- *relative periods* $\Omega^{\varepsilon}(\pi)$: compare bottom-degree cohomological models determined by ε and $-\varepsilon$ (if n is even).

(ε denotes a character of $\pi_0(\mathrm{GL}_n(F \otimes_{\mathbb{Q}} \mathbb{R}))$ which is permissible for π.)

Algebraicity results for special L-values

Fundamental results

- ([Harder, 1983], [Raghuram, 2010], [Raghuram, 2016], [Grobner–Harris, 2016], [Li–Liu–Sun, 2021], among many others) When $m = n-1$ and the coefficient modules of π and π' intertwine, one can relate $L(s_0, \pi \times \pi')$ to $p^{\varepsilon}(\pi) \cdot p^{\varepsilon'}(\pi')$ for suitable ε, ε'.
- ([Harder–Raghuram, 2020]) When (say) m is even, n is odd, and F is totally real, one can relate $\frac{L(s_0, \pi \times \pi')}{L(s_0+1, \pi \times \pi')}$ to $\Omega^{\varepsilon}(\pi')$.

Results of my PhD thesis *(building on [Li–Liu–Sun, 2021], [Harder–Raghuram, 2020])*

Let F be totally real, n be odd, $m < n$ be even.

1. There exist cuspidal representations $\sigma_1, \ldots, \sigma_{(n-m-1)/2}$ of GL_2 such that $\pi'' := \pi' \boxplus \sigma_1 \boxplus \cdots \boxplus \sigma_{(n-m-1)/2}$ is again regular algebraic.
2. If the coefficient modules of π and π'' intertwine and s_0 is critical for $L(s_0, \pi \times \pi')$ and $L(s_0, \pi \times \sigma_j)$ for each j, then
$$L(s_0, \pi \times \pi') \sim i^d \, p^{\varepsilon}(\pi) \, p^{\varepsilon'}(\pi'') \, \mathcal{G}(\omega_{\pi''}) \prod_{j=1}^{(n-m-1)/2} L(s_0, \pi \times \sigma_j)^{-1}.$$
3. (set $m = n-1$) $\quad \Omega^{\varepsilon}(\pi') \sim i^{[F:\mathbb{Q}](n-1)/2} p^{\varepsilon}(\pi') / p^{-\varepsilon}(\pi')$.